Suggested Solutions to:
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Industrial Organization
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Question 1: Vertically related firms and RPM

Part (a)

We can solve for the subgame perfect equilibrium with the help of backward induction. We thus begin by solving the downstream firm's problem. From the question, the downstream firm's profit equals

$$\pi^D = (1-p)(p-w)\frac{e}{a+e} - e.$$

The first-order condition w.r.t. p can be written as

$$\frac{\partial \pi^D}{\partial p} = \left[-(p-w) + (1-p)\right]\frac{e}{a+e} = 0$$
$$\Rightarrow \widehat{p} = \frac{1+w}{2}. \quad (1)$$

The first-order condition w.r.t. e is given by

$$\frac{\partial \pi^D}{\partial e} = (1-p)(p-w)\frac{a}{(a+e)^2} - 1 = 0$$

or, equivalently (using (1)),

$$(a+\widehat{e})^2 = a(1-\widehat{p})\,\widehat{p} - w) = a\left(\frac{1-w}{2}\right)^2$$
$$\Rightarrow \widehat{e} = \sqrt{a}\left(\frac{1-w}{2}\right) - a. \quad (2)$$

In the last step above, we can safely ignore the negative root as we know that $e \geq 0$. For later use, note that

$$\frac{\widehat{e}}{a+\widehat{e}} = \frac{\sqrt{a}\left(\frac{1-w}{2}\right) - a}{\sqrt{a}\left(\frac{1-w}{2}\right)} = 1 - \frac{2\sqrt{a}}{1-w}. \quad (3)$$

Next, consider the first stage, where the upstream firm chooses w. From the question, the upstream firm's profit is

$$\begin{aligned}
\pi^U &= (1-\widehat{p})\frac{\widehat{e}}{a+\widehat{e}}w \\
&= \left(\frac{1-w}{2}\right)\left[1 - \frac{2\sqrt{a}}{1-w}\right]w \\
&= \frac{w\left(1-2\sqrt{a}-w\right)}{2},
\end{aligned}$$

where the second line uses (1) and (3).

The upstream firm's first-order condition can be written as

$$\frac{\partial \pi^U}{\partial w} = \frac{1-2\sqrt{a}-2w}{2} = 0$$
$$\Rightarrow w^* = \frac{1-2\sqrt{a}}{2}. \quad (4)$$

By the assumption $a < \frac{1}{4}$, this expression for w^* is strictly positive. This means that the optimal wholesale price is not at a corner solution, which we implicitly assumed when formulating the first-order condition with an equality. Plugging (4) back into (1) and (2), we obtain

$$p^* = \frac{1+w^*}{2} = \frac{1}{2}\left[1 + \frac{1-2\sqrt{a}}{2}\right] = \frac{3-2\sqrt{a}}{4}$$

and

$$\begin{aligned}
e^* &= \sqrt{a}\left(\frac{1-w^*}{2}\right) - a \\
&= \frac{\sqrt{a}}{2}\left(1 - \frac{1-2\sqrt{a}}{2}\right) - a \\
&= \frac{\sqrt{a}}{4}\left(1+2\sqrt{a}\right) - a \\
&= \frac{\sqrt{a}\left(1-2\sqrt{a}\right)}{4}.
\end{aligned}$$

We can again note that the derived expressions are both positive, thanks to the assumption that $a < \frac{1}{4}$.

Summing up, we have that the equilibrium values of p and e are given by

$$p^* = \frac{3 - 2\sqrt{a}}{4}, \qquad e^* = \frac{\sqrt{a}\,(1 - 2\sqrt{a})}{4}.$$

Part (b)

A simple comparison tells us that $p^* > p^I$ and $e^* < e^I$. The logic behind these relationships is that there is a positive externality between the firms, which is not taken into account when the downstream firm is a separate firm. In particular, both a lower consumer price and a larger effort level increase trade, which has a positive impact (also) on the upstream firm's profit. Therefore, when the firms are integrated, they will choose a lower price and a higher effort.

We should expect consumer surplus (in expected terms) to be larger if the price is lower (for then demand is larger, given a high demand realization) and if the effort is higher (for then the probability of a high demand state is larger). We saw that we indeed have both $p^I < p^*$ and $e^I > e^*$. Therefore, integration should yield the largest (expected) consumer surplus.

Part (c)

Again, we can solve for the subgame perfect equilibrium by using backward induction, beginning with the downstream firm's problem. From the question, the downstream firm's profit equals

$$\pi^D = (1 - p)\,(p - w)\,\frac{e}{a + e} - e.$$

The first-order condition w.r.t. e is given by

$$\frac{\partial \pi^D}{\partial e} = (1 - p)\,(p - w)\,\frac{a}{(a + e)^2} - 1 = 0$$

or, equivalently,

$$(a + \widehat{e})^2 = a\,(1 - p)\,(p - w)$$
$$\Rightarrow \widehat{e} = \sqrt{a\,(1 - p)\,(p - w)} - a. \qquad (5)$$

As in part (a), the negative root is not relevant as we know that $e \geq 0$. For later use, note that

$$\frac{\widehat{e}}{a + \widehat{e}} = \frac{\sqrt{a\,(1 - p)\,(p - w)} - a}{\sqrt{a\,(1 - p)\,(p - w)}} = 1 - \frac{\sqrt{a}}{\sqrt{(1 - p)\,(p - w)}}. \qquad (6)$$

Next, consider the first stage, where the upstream firm chooses w and p. From the question, the upstream firm's profit is

$$\begin{aligned}
\pi^U &= (1 - p)\,\frac{\widehat{e}}{a + \widehat{e}} w \\
&= (1 - p)\left[1 - \frac{\sqrt{a}}{\sqrt{(1 - p)\,(p - w)}}\right] w \\
&= (1 - p)\,w - \sqrt{a}\frac{\sqrt{1 - p}}{\sqrt{p - w}} w \\
&= (1 - p)\,w - \sqrt{a}\,(1 - p)^{\frac{1}{2}}\,(p - w)^{-\frac{1}{2}}\,w,
\end{aligned}$$

where the second line uses (6). The upstream firm's first-order condition w.r.t. w can be written as

$$\frac{\partial \pi^U}{\partial w} = (1 - p) - \sqrt{a}\,(1 - p)^{\frac{1}{2}} \times \left[\frac{1}{2}\,(p - w)^{-\frac{3}{2}}\,w + (p - w)^{-\frac{1}{2}}\right] = 0 \Leftrightarrow$$
$$(1 - p)^{\frac{1}{2}} = \sqrt{a}\,(p - w)^{-\frac{3}{2}}\left[\frac{1}{2} w + (p - w)\right]$$
$$\Leftrightarrow 2\,(1 - p)^{\frac{1}{2}}\,(p - w)^{\frac{3}{2}} = \sqrt{a}\,(2p - w)\,. \qquad (7)$$

Similarly, the first-order condition w.r.t. p can be written as

$$\frac{\partial \pi^U}{\partial p} = -w - \sqrt{a} w \times \left[-\frac{1}{2}\,(1 - p)^{-\frac{1}{2}}\,(p - w)^{-\frac{1}{2}} - \frac{1}{2}\,(1 - p)^{\frac{1}{2}}\,(p - w)^{-\frac{3}{2}}\right] = 0$$
$$\Leftrightarrow 1 = \frac{\sqrt{a}}{2}\,(1 - p)^{-\frac{1}{2}}\,(p - w)^{-\frac{3}{2}}\,[(p - w) + (1 - p)]$$
$$\Leftrightarrow 2\,(1 - p)^{\frac{1}{2}}\,(p - w)^{\frac{3}{2}} = \sqrt{a}\,(1 - w)\,. \qquad (8)$$

Combining (7) and (8), we have

$$\sqrt{a}\,(2p - w) = \sqrt{a}\,(1 - w) \Rightarrow p^R = \frac{1}{2}.$$

The equilibrium price in this model with resale price maintenance is therefore the same as the price under integration, which was stated in the question.

In summary,

$$p^R = \frac{1}{2}, \qquad p^R = p^I$$

In order to answer the last part of the question, plug $p^R = \frac{1}{2}$ into (5) to obtain

$$\begin{aligned}
e^R &= \sqrt{a\,(1 - p^R)\,(p^R - w^R)} - a \\
&= \frac{1}{2}\sqrt{a\,(1 - 2w^R)} - a \\
&= \frac{\sqrt{a}\left(\sqrt{1 - 2w^R} - \sqrt{a}\right)}{2}.
\end{aligned}$$

The equilibrium value of w^R is implicitly defined by (7), evaluated at $w = w^R$ and $p = p^R$:

$$2\left(1-p^R\right)^{\frac{1}{2}}\left(p^R-w^R\right)^{\frac{3}{2}} = \sqrt{a}\left(2p-w^R\right) \Leftrightarrow$$
$$2\left(\frac{1}{2}\right)^{\frac{1}{2}}\left(\frac{1}{2}-w^R\right)^{\frac{3}{2}} = \sqrt{a}\left(1-w^R\right) \Leftrightarrow$$
$$\left(1-2w^R\right)^{\frac{3}{2}} = 2\sqrt{a}\left(1-w^R\right) \Leftrightarrow \varphi\left(\widehat{w}\right)=0,$$
$$\text{where } \varphi\left(w\right) \overset{\text{def}}{=} \left(1-2w\right)^{\frac{3}{2}} - 2\sqrt{a}\left(1-w\right).$$

The function $\varphi(w)$ satisfies $\varphi(0) > 0$, $\varphi\left(\frac{1}{2}\right) < 0$, $\varphi'(0) < 0$, and $\varphi'\left(\frac{1}{2}\right) > 0$. Moreover, it is convex on the interval $[0, \frac{1}{2}]$. (You may want to draw a figure to illustrate this.) It follows that w^R is the unique value of w where the graph of $\varphi(w)$ crosses the horizontal axis in the figure from above.

The last part of the question concerns the relationship between e^R and e^I. We have

$$e^R < e^I \Leftrightarrow \frac{\sqrt{a}\left(\sqrt{1-2w^R}-\sqrt{a}\right)}{2} < \frac{\sqrt{a}\left(1-2\sqrt{a}\right)}{2}$$
$$\Leftrightarrow \sqrt{1-2w^R} < 1-\sqrt{a}$$
$$\Leftrightarrow w^R > \frac{1-\left(1-\sqrt{a}\right)^2}{2} = \frac{\sqrt{a}\left(2-\sqrt{a}\right)}{2}. \quad (9)$$

By using the characterization of w^R above and referring to the figure, we obtain the result that $e^R < e^I$ if and only if the graph of $\varphi(w)$, evaluated at the cutoff in (9), lies above the horizontal axis. That is,

$$e^R < e^I \Leftrightarrow \varphi\left(\frac{\sqrt{a}\left(2-\sqrt{a}\right)}{2}\right) > 0 \Leftrightarrow$$
$$\left(1-2\frac{\sqrt{a}\left(2-\sqrt{a}\right)}{2}\right)^{\frac{3}{2}} > 2\sqrt{a}\left(1-\frac{\sqrt{a}\left(2-\sqrt{a}\right)}{2}\right)$$
$$\Leftrightarrow \left(1-\sqrt{a}\right)^3 > \sqrt{a}\left(2-2\sqrt{a}+a\right). \quad (10)$$

However, it is easy to see that the inequality in (10) is satisfied for $a = 0$, but it is violated (indeed, holds with the opposite inequality) for $a = 1/4$. It follows that it is alternative (iv) in the question that is true:

Whether e^R is smaller or larger than e^I depends on the value of a.

Question 2: Strategic delegation

Part (a)

The game consists of two stages. At the first stage the owners, independently and simultaneously, choose an instruction P_i or R_i. At the second stage we have four different possibilities, depending on what instructions the owners have chosen: both firms are profit maximizers, (P_1, P_2); both firms are revenue maximizers, (R_1, R_2); or one is a profit maximizer and the other is a revenue maximizer, (P_1, R_2) or (R_1, P_2). Given these objectives, the managers choose, independently and simultaneously, a quantity q_i.

- We can solve for the subgame-perfect Nash equilibria of the model by backward induction. We therefore start by solving the four second-stage subgames.

- **The case (P_1, P_2).** Each firm maximizes

$$\begin{aligned} & [45-9(q_1+q_2)]\,q_i - 9q_i \\ = \; & [36-9(q_1+q_2)]\,q_i. \end{aligned}$$

The FOCs for the two firms are

$$-9q_1 + [36-9(q_1+q_2)] = 0,$$
$$-9q_2 + [36-9(q_1+q_2)] = 0.$$

Solving these equations for q_1 and q_2 yields

$$\left(q_1^{PP}, q_2^{PP}\right) = \left(\frac{4}{3}, \frac{4}{3}\right).$$

The profit levels given these outputs are

$$\pi_1^{PP} = \left[45-9\left(q_1^{PP}+q_2^{PP}\right)\right]q_1^{PP} - 9q_1^{PP} = 16,$$
$$\pi_2^{PP} = \left[45-9\left(q_1^{PP}+q_2^{PP}\right)\right]q_2^{PP} - 9q_2^{PP} = 16.$$

- **The case (R_1, R_2).** Each firm maximizes its revenues

$$[45-9(q_1+q_2)]\,q_i.$$

The FOCs for the two firms are

$$-9q_1 + [45-9(q_1+q_2)] = 0,$$
$$-9q_2 + [45-9(q_1+q_2)] = 0.$$

Solving these equations for q_1 and q_2 yields

$$\left(q_1^{RR}, q_2^{RR}\right) = \left(\frac{5}{3}, \frac{5}{3}\right).$$

The profit levels given these outputs are

$$\pi_1^{RR} = \left[45-9\left(q_1^{RR}+q_2^{RR}\right)\right]q_1^{RR} - 9q_1^{RR} = 10,$$
$$\pi_2^{RR} = \left[45-9\left(q_1^{RR}+q_2^{RR}\right)\right]q_2^{RR} - 9q_2^{RR} = 10.$$

- **The case** (P_1, R_2)**.** Firm 1 maximizes its profit

$$\begin{aligned} & [45 - 9(q_1 + q_2)]\, q_i - 9q_i \\ = \; & [36 - 9(q_1 + q_2)]\, q_i. \end{aligned}$$

Firm 1's FOC is

$$-9q_1 + [36 - 9(q_1 + q_2)] = 0. \qquad (11)$$

Firm 2 maximizes its revenues

$$[45 - 9(q_1 + q_2)]\, q_i.$$

Firm 2's FOC is

$$-9q_2 + [45 - 9(q_1 + q_2)] = 0. \qquad (12)$$

Solving equations (11) and (12) for q_1 and q_2 yields

$$\left(q_1^{PR}, q_2^{PR}\right) = (1, 2).$$

The profit levels given these outputs are

$$\pi_1^{PR} = \left[45 - 9\left(q_1^{PR} + q_2^{PR}\right)\right] q_1^{PR} - 9q_1^{PR} = 9$$

and

$$\pi_2^{PR} = \left[45 - 9\left(q_1^{PR} + q_2^{PR}\right)\right] q_2^{PR} - 9q_2^{PR} = 18.$$

- **The case** (R_1, P_2)**.** This is symmetric to the case (P_1, R_2). Therefore, $\left(q_1^{RP}, q_2^{RP}\right) = (2, 1)$,

$$\pi_1^{RP} = 18,$$

and

$$\pi_2^{RP} = 9.$$

- We have now solved all the stage 2 subgames and derived expressions for the equilibrium profit levels in all of these. Using these profit levels we can illustrate the stage 1 interaction between O_1 and O_2 in a game matrix (where O_1 is the row player and O_2 is the column player):

	P_2	R_2
P_1	16, 16	9, 18
R_1	18, 9	10, 10

We see that each player has a strictly dominant strategy and that, in particular, the unique Nash equilibrium of the stage 1 game is that both owners choose revenue maximization, (R_1, R_2).

- **Conclusion:** the game has a unique SPNE. In this equilibrium, both owners choose revenue maximization, (R_1, R_2). In the stage 2 equilibrium path subgame, the managers choose $\left(q_1^{RR}, q_2^{RR}\right) = \left(\frac{5}{3}, \frac{5}{3}\right)$. In the three off-the-equilibrium path subgames, the managers choose $\left(q_1^{PP}, q_2^{PP}\right) = \left(\frac{4}{3}, \frac{4}{3}\right)$, $\left(q_1^{PR}, q_2^{PR}\right) = (1, 2)$, and $\left(q_1^{RP}, q_2^{RP}\right) = (2, 1)$.

Part (b)

Interpretation: The owners would be better off if they both chose to instruct their manager to maximize profit. The reason why this cannot be part of an equilibrium is that each firm can gain by unilaterally instruct its own manager to maximize revenues instead. Why is this the case? First, a manager who maximizes revenues will be more aggressive (i.e., produce more) than a profit maximizing manager. Second, the rival manager, expecting this behavior, will respond by producing less (since the firms' outputs are strategic substitutes). This will increase the first firm's market share and profit.

- If the managers' choice variables had been strategic complements instead we should expect the opposite result: each firm would like to make the rival behave in a way that is good for the own profits (i.e., charge a high price or choose a small quantity). If the choice variables are strategic complements, this means that to induce the rival to behave like that a firm should behave in the same way itself (i.e., charge a high price or choose a small quantity). Therefore, an owner could gain by instructing its manager to be relatively non-aggressive (i.e., to have a strong incentive to charge a high price or choose a small quantity) — this can be achieved by instructing the manager to maximize profits rather than revenues.

- The assumption that the instruction is observable by the rival firm is crucial. Without that assumption, an owner would always want the own manager to maximize profits (but maybe still be *telling* the rival manager that the instruction was R). The point with choosing R is that then the rival *knows* this (and knows that this choice is irreversible), which will (in the model with strategic substitutes) have a beneficial effect on the rival manager's optimal choice at the second stage.